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(C, 1) Means of Orthonormal Expansions for Exponential Weights

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Let $s_m[f]$ denote the mth partial sum of the orthonormal expansion of $f: \mathbb{R} \to \mathbb{R}$ with respect to the orthonormal polynomials for the weight $W^2(x) = \exp(-|x|^\alpha)$, $\alpha > 1$. We show that for some C independent of f and n,

$$\left\| \left(\frac{1}{n} \sum_{m=1}^{n} s_m[f] \right) W \phi_n^{-2/3} \right\|_{L_\infty(\mathbb{R})} \leqslant C \, \|fW\|_{L_\infty(\mathbb{R})}$$

where

$$\phi_n(x) := \left(\left| 1 - \left| \frac{x}{a_n} \right| \right| + n^{-2/3} \right)$$

and a_n denotes the nth Mhaskar–Rahmanov–Saff number for $Q(x) = \frac{1}{2}|x|^\alpha$. The novelty is the presence of the factor $\phi_n^{-2/3}$, which is large close to $\pm a_n$: that factor was absent in the classic results of G. Freud. Related results are proved for more general exponential weights on $(-1, 1)$ or $\mathbb{R}$. © 2000 Academic Press

1. INTRODUCTION AND RESULTS

Let I denote either $(-1, 1)$ or $\mathbb{R}$. Let $W: I \to (0, \infty)$ be such that all the power moments

$$\int_I x^n W^2(x) \, dx, \qquad n \geqslant 0,$$

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are finite. Then we may define orthonormal polynomials

$$p_n(x) = \gamma_n x^n + \cdots, \qquad \gamma_n > 0, \quad n \geqslant 0,$$

satisfying

$$\int_I p_n p_m W^2 = \delta_{mn}.$$

For $f\colon I \to \mathbb{R}$ such that $f(x)\, x^j W^2(x) \in L_1(I)$, $j \geqslant 0$, we may form the formal orthonormal expansion

$$f \leftrightarrow \sum_{j=0}^{\infty} b_j p_j,$$

where

$$b_j := b_j(f) := \int_I f p_j W^2, \qquad j \geqslant 0. \tag{1}$$

The mth partial sum of this expansion is denoted by

$$s_m[f] := \sum_{j=0}^{m-1} b_j(f)\, p_j, \qquad m \geqslant 1. \tag{2}$$

A classic result of G. Freud, proved using the still more classic de la Vallee Poussin argument, asserts that for a class of weights including the exponential weights

$$(W(x) =)\ W_\alpha(x) := \exp(-\tfrac{1}{2}|x|^\alpha), \qquad \alpha > 1, \tag{3}$$

there is strong $(C, 1)$ summability of the orthonormal expansions:

$$\left\| \left(\frac{1}{n} \sum_{m=1}^{n} |s_m[f]| \right) W_\alpha \right\|_{L_\infty(\mathbb{R})} \leqslant C\, \|f W_\alpha\|_{L_\infty(\mathbb{R})} \tag{4}$$

with C independent of f and n. This inequality was the basis of Freud's methods for proving weighted Jackson theorems, see [6, 7, 23]. Strictly speaking Freud considered only $\alpha \geqslant 2$, but later work established that his proofs could be extended to all $\alpha > 1$. For $\alpha < 1$, the polynomials are not dense in a suitable weighted space, while the boundary case $\alpha = 1$ is not fully understood as regards Jackson theorems. See [19, 23] for further orientation.

In this paper, we show that it is possible to strengthen (4) in the sense that one can insert a factor that is large near the largest zero of p_n in the left-hand side of (4). To further elucidate this, we require the notion of the Mhaskar–Rahmanov–Saff number. We shall assume throughout that our weight has the form

$$W = e^{-Q}, \tag{5}$$

where $Q: I \to \mathbb{R}$ is even and convex. The nth Mhaskar–Rahmanov–Saff number a_n is the positive root of the equation

$$n = \frac{2}{\pi} \int_0^1 a_n t Q'(a_n t) \frac{dt}{\sqrt{1-t^2}}, \qquad n \geqslant 1. \tag{6}$$

One of its properties is that

$$\|PW\|_{L_\infty(I)} = \|PW\|_{L_\infty(-a_n, a_n)}, \qquad P \in \mathscr{P}_n, \tag{7}$$

where $\mathscr{P}_n$ denotes the polynomials of degree $\leqslant n$. For example, for $W = W_\alpha$, it is easily seen that

$$a_n = Cn^{1/\alpha}, \qquad n \geqslant 1,$$

where C may be expressed in terms of gamma functions (see [19, 20, 26]).

We shall show that one may insert a factor $(|1 - (|x|/a_n)| + n^{-2/3})^{-1/3}$ in the left-hand side of (4), for a class of weights including W_α, $\alpha > 1$; moreover, when we drop the absolute value in (4), that is when we consider ordinary $(C, 1)$ summability, then we may replace $-1/3$ by $-2/3$. The most general class of Freud weights that we have in mind is given in:

DEFINITION 1. Freud Weights $\mathscr{F}$.

Let $W = e^{-Q}$, where $Q: \mathbb{R} \to \mathbb{R}$ is even, continuous and Q'' is continuous in $(0, \infty)$. Assume moreover, that $Q' > 0$ in $(0, \infty)$, and that for some A, $B > 1$,

$$A \leqslant 1 + \frac{xQ''(x)}{Q'(x)} \leqslant B, \qquad x \in (0, \infty). \tag{8}$$

Then we write $W \in \mathscr{F}$.

Note that for $W = W_\alpha$, (8) holds with $A = B = \alpha$. In addition to Freud weights on the real line, we consider a class of Erdős weights, for which the exponent Q grows faster than any polynomial:

Definition 2. Erdős Weights $\mathscr{E}$.
Let $W = e^{-Q}$, where $Q: \mathbb{R} \to \mathbb{R}$ is even, continuous and Q'' is continuous in $(0, \infty)$. Assume that $Q' > 0$, $Q'' \geqslant 0$ in $(0, \infty)$, and that the function

$$T(x) := 1 + \frac{xQ''(x)}{Q'(x)}, \qquad x \in (0, \infty) \tag{9}$$

is increasing in $(0, \infty)$ with

$$\lim_{x \to 0+} T(x) > 1; \qquad \lim_{x \to \infty} T(x) = \infty. \tag{10}$$

Assume moreover that for some $C_j > 0$, $j = 1, 2, 3$,

$$C_1 \leqslant T(x) \frac{Q(x)}{xQ'(x)} \leqslant C_2, \qquad x \geqslant C_3.$$

Then we write $W \in \mathscr{E}$.

The archetypal example of $W \in \mathscr{E}$ is

$$W(x) = W_{k, \alpha}(x) = \exp(-\exp_k(|x|^\alpha)) \tag{11}$$

where $\alpha > 1$ and $k \geqslant 1$ and

$$\exp_k := \underbrace{\exp(\exp(\cdots \exp() \cdots))}_{k \text{ times}}$$

denotes the kth iterated exponential. We also set

$$\exp_0(x) := x.$$

See [12, 13] for further orientation on Erdős weights.
The third class of weights we consider is a class of exponential weights on $(-1, 1)$:

Definition 3. Exponential Weights on $(-1, 1)$ $\mathscr{EXP}$.
Let $W = e^{-Q}$, where $Q: (-1, 1) \to \mathbb{R}$ is even and Q'' is continuous in $(-1, 1)$. Assume that $Q' > 0$, $Q'' \geqslant 0$ in $(0, 1)$, and that the function

$$T(x) := 1 + \frac{xQ''(x)}{Q'(x)}, \qquad x \in (0, 1) \tag{12}$$

is increasing in $(0, 1)$ with

$$\lim_{x \to 0+} T(x) > 1. \tag{13}$$

Assume moreover that for some $C_1>0$, $C_2>0$,

$$C_1 \leqslant T(x)\,\frac{Q(x)}{Q'(x)} \leqslant C_2, \qquad x \text{ close enough to } 1 \tag{14}$$

and that for some $A>2$ and x close enough to 1,

$$T(x) \geqslant \frac{A}{1-x^2}. \tag{15}$$

Then we write $W \in \mathscr{EXP}$.

The archetypal example of $W \in \mathscr{EXP}$ is

$$W(x) = W^{k,\alpha}(x) := \exp(-\exp_k((1-x^2)^{-\alpha})), \qquad x \in (-1, 1) \tag{16}$$

where $k \geqslant 0$, $\alpha > 0$. For further orientation on $\mathscr{EXP}$, see [10].

It is possible to treat the classes $\mathscr{F}$, $\mathscr{E}$, $\mathscr{EXP}$ in a more general and unified framework [11], but we prefer here to quote already published results. In any event, it is possible to describe simultaneously several features of the $(C, 1)$ means of the orthonormal expansions for all three classes of weights: this requires some additional notation. We set

$$\delta_n := (nT(a_n))^{-2/3}, \qquad n \geqslant 1, \tag{17}$$

and define the functions

$$\phi_n(x) := \left|1 - \frac{|x|}{a_n}\right| + \delta_n \tag{18}$$

and

$$\psi_n(x) := \frac{\phi_n(x) + T(a_n)^{-1}}{\sqrt{\phi_n(x)}} = \frac{\left|1 - \frac{|x|}{a_n}\right| + \delta_n + T(a_n)^{-1}}{\sqrt{\left|1 - \frac{|x|}{a_n}\right| + \delta_n}}. \tag{19}$$

The function ψ_n plays a role in describing the spacing between successive zeros of p_n, the growth of Christoffel functions, and related quantities, in much the same way as does the function $1-x^2+n^{-2}$ for Jacobi weights and their generalizations on $(-1, 1)$. Note that for Freud weights, T is bounded above and below by positive constants, so δ_n behaves like $n^{-2/3}$.

By a minor modification of the classical de la Vallee Poussin argument for L_∞ and then via standard duality and interpolation techniques, we prove:

THEOREM 4. *Let $W \in \mathcal{F}$, $\mathcal{E}$ or $\mathcal{EXP}$. Let $1 \leqslant p < \infty$ and let*

$$\Psi_n(x) := \max\{\psi_n^{1/2}(x), \psi_n^{2/3}(x)\}, \qquad x \in I. \tag{20}$$

Then for some C independent of f and n,

$$\left\| \left(\frac{1}{n} \sum_{m=1}^{n} s_m[f] \right) W \Psi_n^{1-1/p} \right\|_{L_p(I)} \leqslant C \, \| f W \Psi_n^{-1/p} \|_{L_p(I)}. \tag{21}$$

For the case $p = \infty$, we have

$$\left\| \left(\frac{1}{n} \sum_{m=1}^{n} |s_m[f]| \right) W \Psi_n \right\|_{L_\infty(I)} \leqslant C \, \| f W \|_{L_\infty(I)}. \tag{22}$$

We note that if one uses the classical de la Vallee Poussin argument, one has to omit the $\psi_n^{2/3}$ in (20); our modification permits the inclusion of this factor.

In [16], strong $(C, 1)$ means of orthonormal expansions for Erdős weights were investigated; there for $p = \infty$, instead of Ψ_n in (22) there was a factor $T^{-1/4}$ in the left-hand side. Since one can show that

$$T^{-1/4} \leqslant C \psi_n^{1/2} \leqslant C \Psi_n$$

for the class $\mathcal{E}$, the above result constitutes an improvement of the result in [16].

To acquire some perspective on how Theorem 4 relates to Freud's (4), we specialize to Freud weights. Here $\Psi_n / \phi_n^{-1/3}$ is bounded above and below by positive constants and we obtain:

COROLLARY 5. *Let $W \in \mathcal{F}$. Let $1 \leqslant p < \infty$. Then for some C independent of f and n,*

$$\left\| \left(\frac{1}{n} \sum_{m=1}^{n} s_m[f] \right) W \phi_n^{-(1-1/p)/3} \right\|_{L_p(I)} \leqslant C \, \| f W \phi_n^{1/(3p)} \|_{L_p(I)}. \tag{23}$$

For the case $p = \infty$, we have

$$\left\| \left(\frac{1}{n} \sum_{m=1}^{n} |s_m[f]| \right) W \phi_n^{-1/3} \right\|_{L_\infty(I)} \leqslant C \, \| f W \|_{L_\infty(I)}. \tag{24}$$

Thus under the same hypotheses as Freud, one may insert the factor $\phi_n^{-1/3}$, which is large near $\pm a_n$. The obvious question is whether or not 1/3 is sharp. If one assumes more about the orthonormal polynomials, it is not.

Recall that the orthonormal polynomials $\{p_n\}$ satisfy the three term recurrence relation

$$xp_{n-1}(x) = \alpha_n p_n(x) + \alpha_{n-1} p_{n-2}(x), \qquad n \geqslant 1 \tag{25}$$

where we set $p_{-1} := 0$ and

$$\alpha_n := \gamma_{n-1}/\gamma_n, \qquad n \geqslant 1. \tag{26}$$

It is known for large classes of Freud weights [14] that

$$\alpha_n = \tfrac{1}{2} a_n (1 + o(1)), \qquad n \to \infty. \tag{27}$$

Assuming somewhat more allows us to improve on the 1/3 in (23) and (24):

THEOREM 6. *Let $W \in \mathscr{F}$ and assume that for some $\beta > 0$,*

$$\alpha_n = \tfrac{1}{2} a_n (1 + O(n^{-\beta})) \qquad n \to \infty. \tag{28}$$

Let

$$\kappa := \min\left\{\frac{2}{3}, \frac{1}{3} + \frac{\beta}{2}, \frac{5}{12} + \frac{\beta}{4}\right\}. \tag{29}$$

Let $1 \leqslant p < \infty$. Then for some C independent of f and n,

$$\left\| \left(\frac{1}{n} \sum_{m=1}^{n} s_m[f] \right) W \phi_n^{-(1-1/p)\kappa} \right\|_{L_p(I)} \leqslant C \, \| f W \phi_n^{\kappa/p} \|_{L_p(I)}. \tag{30}$$

For the case $p = \infty$, we have

$$\left\| \left(\frac{1}{n} \sum_{m=1}^{n} s_m[f] \right) W \phi_n^{-\kappa} \right\|_{L_\infty(I)} \leqslant C \, \| f W \|_{L_\infty(I)}. \tag{31}$$

For W_α, $\alpha > 1$, (28) is known with $\beta = \min\{\alpha, 2\}$. This was recently proved by Kriecherbauer and McLaughlin [8], thereby improving results of Rakhmanov [25]. For α a positive even integer, more complete asymptotics are known, [3], [18]. Likewise when Q is a polynomial, more complete asymptotics are known [1, 3]. Thus we may deduce:

COROLLARY 7. *For $W = W_\alpha$, $\alpha > 1$, and $1 \leqslant p < \infty$,*

$$\left\| \left(\frac{1}{n} \sum_{m=1}^{n} s_m[f] \right) W_\alpha \phi_n^{-(2/3)(1-1/p)} \right\|_{L_p(I)} \leqslant C \, \| f W_\alpha \phi_n^{2/3p} \|_{L_p(I)}. \tag{32}$$

For the case $p=\infty$, we have

$$\left\| \left(\frac{1}{n} \sum_{m=1}^{n} s_m[f] \right) W_\alpha \phi_n^{-2/3} \right\|_{L_\infty(I)} \leqslant C \, \| f W_\alpha \|_{L_\infty(I)}. \tag{33}$$

It is an interesting problem to determine the sharp power of ϕ_n in (33).

This paper is organised as follows: in Section 2, we present the de la Vallee Poussin argument, and its minor modification, which leads to the proof of Theorem 4 and hence Corollary 5. In Section 3, we present an estimate on sums of squares of $p_{m+1}-p_{m-1}$, under the assumption (28). In Section 4, we prove Theorem 6 and deduce Corollary 7.

2. PROOF OF THEOREM 4

We begin by recalling the classic de la Vallee Poussin argument. (This has been clearly presented often [7], [23],... but we do need the details). Let $f: I \to \mathbb{R}$, $x \in I$ and $\rho_n > 0$. We let

$$f_n(t) := \begin{cases} f(t), & |t-x| \leqslant \rho_n \\ 0, & |t-x| > \rho_n \end{cases} \tag{34}$$

and

$$F_n(t) := \frac{f(t)-f_n(t)}{x-t} = \begin{cases} \dfrac{f(t)}{x-t}, & |t-x| > \rho_n \\ 0, & |t-x| \leqslant \rho_n. \end{cases} \tag{35}$$

Then we may split for $m \leqslant n$,

$$s_m[f](x) = s_m[f_n](x) + s_n[F_n(\cdot)(x-\cdot)](x). \tag{36}$$

Let

$$K_m(x,t) := \sum_{j=0}^{m-1} p_j(x)\, p_j(t) \tag{37}$$

so that

$$s_m[f_n](x) = \int_I K_m(x,t)\, f_n(t)\, W^2(t)\, dt. \tag{38}$$

The de la Vallee Poussin/Freud Estimate for $s_m[f_n](x)$.

$$|s_m[f_n](x)| \leqslant \|fW\|_{L_\infty(I)} \int_{I \cap [x-\rho_n, x+\rho_n]} |K_m(x, t)|\, W(t)\, dt \tag{39}$$

$$\leqslant \|fW\|_{L_\infty(I)} \sqrt{2\rho_n} \sqrt{\int_I K_m^2(x, t)\, W^2(t)\, dt}$$

$$= \|fW\|_{L_\infty(I)} \sqrt{2\rho_n} \sqrt{\sum_{j=0}^{m-1} p_j^2(x)}, \tag{40}$$

by the Cauchy–Schwarz inequality and then orthogonality. Recall now the Christoffel function:

$$\lambda_m^{-1}(W^2, x) := \sum_{j=0}^{m-1} p_j^2(x). \tag{41}$$

Since λ_m^{-1} clearly increases with m, we deduce that (note that the λ_{n+1} simplifies later calculations)

$$\frac{1}{n} \sum_{m=1}^{n} |s_m[f_n](x)|\, W(x) \leqslant \|fW\|_{L_\infty(I)} \sqrt{2\rho_n} \sqrt{\lambda_{n+1}^{-1}(W^2, x)\, W^2(x)}. \tag{42}$$

The de la Vallee Poussin/Freud Estimate for $s_m[F_n(\cdot)(x-\cdot)](x)$. We need the Christoffel Darboux formula

$$K_m(x, t) = \alpha_m \frac{p_m(x)\, p_{m-1}(t) - p_{m-1}(x)\, p_m(t)}{x-t}. \tag{43}$$

We see then that

$$s_m[F_n(\cdot)(x-\cdot)](x) = \int_I K_m(x, t)\, F_n(t)(x-t)\, W^2(t)\, dt$$

$$= \alpha_m [p_m(x)\, b_{m-1}(F_n) - p_{m-1}(x)\, b_m(F_n)]. \tag{44}$$

Let us abbreviate $b_m(F_n)$ as b_m. We deduce that

$$\frac{1}{n} \sum_{m=1}^{n} |s_m[F_n(\cdot)(x-\cdot)(x)]|$$

$$\leqslant \frac{1}{n} (\max_{1 \leqslant m \leqslant n} \alpha_m) \sum_{m=1}^{n} (|p_m(x)|\, |b_{m-1}| + |p_{m-1}(x)|\, |b_m|)$$

$$\leqslant (\max_{1 \leqslant m \leqslant n} \alpha_m) \frac{2}{n} \sqrt{\sum_{m=0}^{n} p_m^2(x)} \sqrt{\sum_{m=0}^{n} b_m^2}$$

by the Cauchy–Schwarz inequality. Using Bessel's inequality for orthonormal expansions, we continue this as

$$\leqslant (\max_{1\leqslant m\leqslant n} \alpha_m) \frac{2}{n} \sqrt{\lambda_{n+1}^{-1}(W^2, x)} \sqrt{\int_I F_n^2 W^2}$$

$$\leqslant (\max_{1\leqslant m\leqslant n} \alpha_m) \frac{2}{n} \sqrt{\lambda_{n+1}^{-1}(W^2, x)} \, \|fW\|_{L_\infty(I)} \sqrt{\int_{|t-x|\geqslant \rho_n} \frac{dt}{(t-x)^2}}$$

$$= (\max_{1\leqslant m\leqslant n} \alpha_m) \frac{2\sqrt{2}}{n} \sqrt{\lambda_{n+1}^{-1}(W^2, x)\, \rho_n^{-1}} \, \|fW\|_{L_\infty(I)}. \tag{45}$$

The de la Vallee Poussin estimate for the strong $(C, 1)$ *means of* $s_m[f]$. Combining (36), (42) and (45) gives

$$\frac{1}{n} \sum_{m=1}^{n} |s_m[f](x)| \, W(x) \leqslant \|fW\|_{L_\infty(I)} \sqrt{\lambda_{n+1}^{-1}(W^2, x)\, W^2(x)}$$
$$\times \left(\sqrt{2\rho_n} + (\max_{1\leqslant m\leqslant n} \alpha_m)\, 2 \sqrt{\frac{2}{n^2 \rho_n}} \right). \tag{46}$$

Choosing

$$\rho_n := \frac{\max_{1\leqslant m\leqslant n} \alpha_m}{n}$$

gives

$$\frac{1}{n} \sum_{m=1}^{n} |s_m[f](x)| \, W(x)$$
$$\leqslant 5 \, \|fW\|_{L_\infty(I)} \sqrt{\lambda_{n+1}^{-1}(W^2, x)\, W^2(x)} \sqrt{\frac{\max_{1\leqslant m\leqslant n} \alpha_m}{n}}. \tag{47}$$

We turn to a minor modification of the de la Vallee Poussin/Freud estimate before proving Theorem 4:

A simple alternative estimate for $s_m[f_n](x)$. Now for $|t-x| \leqslant \rho_n$, and $m \leqslant n$, the Cauchy–Schwarz inequality gives

$$\begin{aligned} |K_m(x, t)| &\leqslant \sqrt{K_m(x, x)} \sqrt{K_m(t, t)} \\ &\leqslant \sqrt{K_{n+1}(x, x)} \sqrt{K_{n+1}(t, t)} \\ &= \sqrt{\lambda_{n+1}^{-1}(W^2, x)} \sqrt{\lambda_{n+1}^{-1}(W^2, t)}. \end{aligned}$$

Then from (39),

$$|s_m[f_n](x)|\, W(x) \leqslant \|fW\|_{L_\infty(I)}\, 2\rho_n \sqrt{\lambda_{n+1}^{-1}(W^2, x)\, W^2(x)} \times \sqrt{\max_{|t-x| \leqslant \rho_n} \lambda_{n+1}^{-1}(W^2, t)\, W^2(t)}. \tag{48}$$

Then instead of (46), we obtain

$$\frac{1}{n} \sum_{m=1}^{n} |s_m[f](x)|\, W(x) \leqslant \|fW\|_{L_\infty(I)} \sqrt{\lambda_{n+1}^{-1}(W^2, x)\, W^2(x)} \times \left\{ 2\rho_n \sqrt{\max_{|t-x| \leqslant \rho_n} \lambda_{n+1}^{-1}(W^2, t)\, W^2(t)} + (\max_{1 \leqslant m \leqslant n} \alpha_m)\, 2 \sqrt{\frac{2}{n^2 \rho_n}} \right\}.$$

Choosing

$$\rho_n := \left(\frac{\max_{1 \leqslant m \leqslant n} \alpha_m}{n} \sqrt{\lambda_{n+1}(W^2, x)/W^2(x)} \right)^{2/3} \tag{49}$$

gives

$$\frac{1}{n} \sum_{m=1}^{n} |s_m[f](x)|\, W(x) \leqslant 3\, \|fW\|_{L_\infty(I)} \left(\frac{\max_{1 \leqslant m \leqslant n} \alpha_m}{n}\, \lambda_{n+1}^{-1}\, (W^2, x)\, W^2(x) \right)^{2/3} \times \left[\max_{|t-x| \leqslant \rho_n} \left(\frac{\lambda_{n+1}^{-1}(W^2, t)\, W^2(t)}{\lambda_{n+1}^{-1}(W^2, x)\, W^2(x)} \right)^{1/2} + 1 \right]. \tag{50}$$

Thus far, we have the estimates (47) and (50) for the strong $(C, 1)$ means. Before we can choose which to apply, we need technical estimates for λ_{n+1}^{-1}, for α_m and so on. We use the standard notation $\sim$ for sequences of real numbers: we write

$$c_n \sim d_n$$

if there exists positive constant C_1, C_2 independent of n such that for the relevant range of n,

$$C_1 \leqslant c_n / d_n \leqslant C_2.$$

Similar notation is used for functions and sequences of functions. Moreover, in the sequel, C, C_1, $C_2, \ldots$ denote positive constants independent of n, x, f. The same symbol does not necessarily denote the same constant in different occurrences.

LEMMA 8. *Let $W \in \mathcal{F}$, $\mathcal{E}$ or $\mathcal{EXP}$. Then*

(a)

$$\max_{1 \leqslant m \leqslant n} \alpha_m \sim \alpha_n \sim a_n. \tag{51}$$

(b) *Let η, $L > 0$. There exists n_0 such that uniformly for $n \geqslant n_0$ and for $|x| \leqslant a_n(1 + L\delta_n)$,*

$$\lambda_n(W^2, x) \sim \frac{a_n}{n} W^2(x)\, \psi_n(x). \tag{52}$$

(c) *Let $\eta > 0$. There exists n_0 such that uniformly for $n \geqslant n_0$ and for $|x| \leqslant a_n(1 + L\delta_n)$,*

$$|t - x| \leqslant \eta \frac{a_n}{n} \psi_n(x) \Rightarrow \psi_n(t) \sim \psi_n(x) \qquad \text{and} \qquad \phi_n(t) \sim \phi_n(x). \tag{53}$$

The constants in $\sim$ are independent of n, x, t.

(d) *There exists n_0 such that for $n \geqslant n_0$*

$$\frac{1}{2} \leqslant \frac{m}{n} \leqslant 2 \Rightarrow \left|1 - \frac{a_m}{a_n}\right| \sim \frac{1}{T(a_n)} \left|1 - \frac{m}{n}\right|. \tag{54}$$

Moreover,

$$T(a_n) \sim T(a_{2n}); \qquad \delta_n \sim \delta_{2n}; \qquad \delta_n^{-1/2} = o(n). \tag{55}$$

(e) *Let $L > 0$. There exists n_0 such that uniformly for $n \geqslant n_0$ and for $|x| \leqslant a_n(1 + L\delta_n)$,*

$$\psi_n(x) \sim \psi_{n+1}(x); \qquad \phi_n(x) \sim \phi_{n+1}(x). \tag{56}$$

(*f*) *Let $L > 0$, $0 < p \leqslant \infty$. There exist C and n_0 such that for $n \geqslant n_0$ and for $P \in \mathcal{P}_n$,*

$$\|PW\|_{L_p(I)} \leqslant C\, \|PW\|_{L_p(-a_n(1 - L\delta_n),\, a_n(1 - L\delta_n))}. \tag{57}$$

Moreover, if $r > 1$, there exist C_1, $C_2 > 0$ such that for $n \geqslant 1$ and for $P \in \mathcal{P}_n$,

$$\|PW\|_{L_p(I \setminus [a_{-rn},\, a_{rn}])} \leqslant C_1 \exp(-C_2 n T(a_n)^{-1/2})\, \|PW\|_{L_p(I)}. \tag{58}$$

Proof. (a) We note that since a_m increases with m, it suffices to show that

$$\alpha_m \sim a_m, \qquad m \geqslant 1.$$

For $W \in \mathscr{F}$, this is Theorem 12.3(b) in [9, p. 529]; for $W \in \mathscr{E}$, this is (10.33) in [12, p. 285]; for $W \in \mathscr{E}\mathscr{X}\mathscr{P}$, this follows from a far more general result of Rakhmanov [24] that for $W > 0$ a.e. in $[-1, 1]$, $\alpha_m \to \frac{1}{2}$, $m \to \infty$.

(b) For $W \in \mathscr{F}$, Theorem 1.1 in [9, p. 465] states that

$$\lambda_n(W^2, x)/W^2(x) \sim \frac{a_n}{n}\, \phi_n^{-1/2}(x) \sim \frac{a_n}{n} \left(\left| 1 - \frac{|x|}{a_n} \right| + n^{-2/3} \right)^{-1/2}$$

for the relevant range of n and x. Note that for $W \in \mathscr{F}$, $A \leqslant T \leqslant B$, where A, B are as in (8), so

$$\psi_n(x) = \frac{\phi_n(x) + T(a_n)^{-1}}{\sqrt{\phi_n(x)}} \sim \frac{1}{\sqrt{\phi_n(x)}}.$$

Thus we have (52) in this case. Next, if $W \in \mathscr{E}$, Theorem 1.2 in [12, p. 204] implies that

$$\lambda_n(W^2, x)/W^2(x) \sim \frac{a_n}{n} \max\{\sqrt{\phi_n(x)}, [T(a_n)\sqrt{\phi_n(x)}]^{-1}\} \tag{59}$$

for the relevant range of n and x. This is easily recast in the form (52). Finally, if $W \in \mathscr{E}\mathscr{X}\mathscr{P}$, Theorem 1.2 in [10, p. 7] again implies (59) and hence (52).

(c) In view of the form of ψ_n, it clearly suffices to show that $\phi_n(t) \sim \phi_n(x)$ for the relevant range n, t, x. Let us denote the zeros of $p_n(x) = p_n(W^2, x)$ by

$$x_{nn} < x_{n-1,n} < \cdots < x_{2n} < x_{1n}.$$

It is known for all three classes of weights that uniformly in n and j,

$$\phi_n(x_{jn}) \sim \phi_n(x_{j-1,n}) \text{ and hence } \psi_n(x_{jn}) \sim \psi_n(x_{j-1,n}). \tag{60}$$

For $W \in \mathscr{F}$, this is (11.10) in [9, p. 521]; for $W \in \mathscr{E}$, this is (9.9) in [12, p. 265]; and for $W \in \mathscr{E}\mathscr{X}\mathscr{P}$, this is (10.12), in [10, p. 111]. Next, for all three classes of weights it is known that uniformly in n and j;

$$\begin{aligned} x_{j-1,n} - x_{j+1,n} &\sim \lambda_{jn} W^{-2}(x_{jn}) \\ &\sim \frac{a_n}{n} \psi_n(x_{jn}); \qquad \left| 1 - \frac{x_{1n}}{a_n} \right| \leqslant C\delta_n. \end{aligned} \tag{61}$$

For $W \in \mathscr{F}$, this follows from (b) above and Corollary 1.2 in [9, pp. 466-467]; for $W \in \mathscr{E}$, this follows from Corollary 1.3 in [12, p. 205]; and for $W \in \mathscr{EXP}$, this follows from Corollary 1.4 in [10, p. 9]. The monotonicity of ϕ_n in $[0, a_n]$ or $[-a_n, 0]$ and (60) and (61) then give the result.

(d) For $W \in \mathscr{F}$, these follow from Lemma 5.2(c) in [9, p. 478] (recall that $T \sim 1$ and $\delta_n \sim n^{-2/3}$ for this case); for $W \in \mathscr{E}$, these follow from Lemma 2.2 in [12, pp. 208–209]; and for $W \in \mathscr{EXP}$, these follow from Lemma 3.2 in [10, p. 24–25].

(e) This follows easily from (d), which shows that

$$\left|1 - \frac{a_n}{a_{n+1}}\right| \sim \frac{1}{nT(a_n)} = o(\delta_n), \qquad n \to \infty.$$

(f) For $W \in \mathscr{F}$, (57) is Theorem 1.8 in [9, p. 469] while (58) follows easily from (7.14) in [9, p. 486] and (10.2) in [9, p. 512]; for $W \in \mathscr{E}$, (57) is Theorem 1.5 in [12, p. 206] while (58) follows from (4.18) in [12, p. 228] and (5.2) in [12, p. 231]; and for $W \in \mathscr{EXP}$, (57) is Theorem 1.7 in [10, p. 12] while (58) follows from (5.18) in [10, p. 53] and (6.2) in [10, p. 57]. ∎

We proceed to:

Proof of Theorem 4 *for* $p = \infty$. Let us substitute the estimates of the last lemma in (47): we obtain for $|x| \leqslant a_n$,

$$\frac{1}{n} \sum_{m=1}^{n} |s_m[f](x)| \; W(x) \leqslant C \, \|fW\|_{L_\infty(I)} \, \psi_n^{-1/2}(x). \tag{62}$$

Next, provided we choose ρ_n by (49), so that by Lemma 8(a), (b), (e),

$$\rho_n \sim \frac{a_n}{n} \psi_n(x)^{1/3} \tag{63}$$

we have also from (50) and Lemma 8(a), (b), (e),

$$\frac{1}{n} \sum_{m=1}^{n} |s_m[f](x)| \; W(x)$$
$$\leqslant C \, \|fW\|_{L_\infty(I)} \, \psi_n^{-2/3}(x) \left[\max_{|t-x| \leqslant \rho_n} \left(\frac{\psi_n(t)}{\psi_n(x)} \right)^{-1/2} + 1 \right].$$

Now if

$$\psi_n(x) \geqslant \tfrac{1}{2},$$

then (63) shows that

$$\rho_n \leqslant C \frac{a_n}{n} \psi_n(x)$$

and then from Lemma 8(c),

$$\max_{|t-x| \leqslant \rho_n} \left(\frac{\psi_n(t)}{\psi_n(x)} \right)^{-1/2} \leqslant C.$$

Thus

$$\psi_n(x) \geqslant \frac{1}{2} \Rightarrow \frac{1}{n} \sum_{m=1}^{n} |s_m[f](x)| \, W(x) \leqslant C \, \|fW\|_{L_\infty(I)} \, \psi_n^{-2/3}(x).$$

This and (62) show that

$$\frac{1}{n} \sum_{m=1}^{n} |s_m[f](x)| \, W(x) \max\{\psi_n^{1/2}, \psi_n^{2/3}\}(x) \leqslant C \, \|fW\|_{L_\infty(I)}.$$

When $\psi_n(x) < \frac{1}{2}$, (62) shows that this inequality persists as then $\psi_n^{2/3}(x) < \psi_n^{1/2}(x)$. Thus

$$\max_{|x| \leqslant a_n} \frac{1}{n} \sum_{m=1}^{n} |s_m[f](x)| \, W(x) \max\{\psi_n^{1/2}, \psi_n^{2/3}\}(x) \leqslant C \, \|fW\|_{L_\infty(I)}. \tag{64}$$

To extend this to the rest of I, we use infinite-finite range inequalities in the following way: let us suppose that there are polynomials R_n with the following properties:

(i) R_n has degree $O(\delta_n^{-1/2})$;

(ii) $R_n \sim \Psi_n = \max\{\psi_n^{1/2}, \psi_n^{2/3}\}$ in $[-a_n, a_n]$;

(iii) $R_n \geqslant C\Psi_n$ in $I \backslash [-a_n, a_n]$.

We now use a device of J. Szabados [27] to apply the infinite-finite range inequalities: for any $\varepsilon_m = \pm 1$, (64) gives

$$\max_{|x| \leqslant a_n} \left| \frac{1}{n} \sum_{m=1}^{n} \varepsilon_m s_m[f](x) \, R_n(x) \right| \, W(x) \leqslant C \, \|fW\|_{L_\infty(I)}.$$

The expression is the $|\ |$ is a polynomial of degree at most $[n + C\delta_n^{-1/2}]$ for some C (here $[x]$ denotes the integer part of x). But by (54) and then the third relation in (55),

$$\left| 1 - \frac{a_n}{a_{[n + C\delta_n^{-1/2}]}} \right| \sim \frac{\delta_n^{-1/2}}{nT(a_n)} = \delta_n \sim \delta_{[n + C\delta_n^{-1/2}]}$$

so for some $L>0$, if n is large enough,

$$\max_{|x| \leqslant a_{[n+C\delta_n^{-1/2}]}(1-L\delta_{[n+C\delta_n^{-1/2}]})} \left|\frac{1}{n} \sum_{m=1}^{n} \varepsilon_m s_m[f](x)\, R_n(x)\right| W(x)$$
$$\leqslant C\, \|fW\|_{L_\infty(I)}.$$

The infinite-finite range inequality (57) shows that as the choice $\varepsilon_m = \pm 1$ is arbitrary,

$$\max_{x\in I} \frac{1}{n} \sum_{m=1}^{n} |s_m[f](x)|\, R_n(x)\, W(x) \leqslant C\, \|fW\|_{L_\infty(I)}.$$

Finally, since $\Psi_n = O(R_n)$ in I, we obtain (22). It remains to give

The Construction of the $\{R_n\}$ *satisfying* (i), (ii), (iii) *above.* Now $\Psi_n \sim \psi_n^{2/3} + \psi_n^{1/2}$, and $\psi_n = \sqrt{\phi_n} + 1/(T(a_n)\sqrt{\phi_n})$ so it suffices to show the following: given $b \in \mathbb{R}$, there exist polynomials R_n^* such that

(i*) R_n^* has degree $O(\delta_n^{-1/2})$;

(ii*) $R_n^* \sim \phi_n^b$ in $[-a_n, a_n]$;

(iii*) $R_n^* \geqslant C\phi_n^b$ in $I \backslash [-a_n, a_n]$.

We need only do this for $|b| < \frac{1}{2}$ (raising to suitable powers gives the general case). We use the Christoffel functions for the ultraspherical weight

$$u(x) := (1-x^2)^{-b-(1/2)}, \qquad x \in (-1, 1).$$

Let us set

$$m := m(n) := [\delta_n^{-1/2}];$$
$$R_n^{\#}(x) := m^{-1} \lambda_m^{-1}(u, x).$$

It is well known that uniformly in m, x [21, p. 120]

$$R_n^{\#}(x) \sim (|1-|x|| + m^{-2})^b \qquad \text{in } [-1, 1]. \tag{65}$$

Then it is easily seen that

$$R_n^*(x) := R_n^{\#}\left(\frac{x}{a_n}\right)$$

satisfies (i*), (ii*). To verify (iii*), it suffices to show that

$$R_n^{\#}(x) \geqslant C(x-1+m^{-2})^b, \qquad x \in (1, \infty).$$

(Recall that $R_n^{\#}$ is even). Let ℓ denote the least integer $\geqslant b/2$. Let $p_j^{\#}$ denote the jth orthonormal polynomial for u, so that its zeros lie in $(-1, 1)$, and for some integer j_0 and $C_1 > 0$, $p_j^{\#}$ has at least ℓ zeros in $[1-(C_1 j)^{-2}, 1]$ for $j \geqslant j_0$. See, for example, [21, Thm. 22, p. 167]. Then for $j \geqslant j_0$ and $x > 1$,

$$\frac{p_j^{\#}(x)}{p_j^{\#}(1)} = \prod_{y:\, p_j^{\#}(y)=0} \left(1+\frac{x-1}{1-y}\right) \geqslant (1+(C_1 j)^2 (x-1))^{\ell}.$$

Let $\eta \in (0, 1)$. Then for $x > 1$, $m \geqslant j_0/\eta$,

$$\begin{aligned}\lambda_m^{-1}(u, x) &> \sum_{j=[\eta m]+1}^{m-1} (p_j^{\#}(x))^2 \\ &\geqslant (1+(C_1 \eta m)^2 (x-1))^b \sum_{j=[\eta m]+1}^{m-1} (p_j^{\#}(1))^2.\end{aligned}$$

It follows easily from the fact that $|b| < \frac{1}{2}$ and from the estimate

$$k^{-1} \lambda_k^{-1}(u, 1) \sim k^{-2b}, \qquad k \geqslant 1,$$

that if η is small enough,

$$\sum_{j=[\eta m]+1}^{m-1} (p_j^{\#}(1))^2 \sim \sum_{j=0}^{m-1} (p_j^{\#}(1))^2 = \lambda_m^{-1}(u, 1) \sim m^{1-2b}$$

and hence that for $x > 1$,

$$\begin{aligned}R_n^{\#}(x) &= m^{-1} \lambda_m^{-1}(u, x) \\ &\geqslant (1+(c_1 \eta m)^2 (x-1))^b\, m^{-2b} \geqslant C(x-1+m^{-2})^b,\end{aligned}$$

as desired. ∎

The extension from $p=\infty$ to $1 \leqslant p < \infty$ is entirely standard [6], but we provide the details:

The proof of Theorem 4 *for* $p=1$. Now

$$\begin{aligned}\left\| \frac{1}{n} \sum_{m=1}^{n} s_m[f]\, W \right\|_{L_1(I)} &= \sup_{\|gW\|_{L_\infty(I)} \leqslant 1} \int_I \left(\frac{1}{n} \sum_{m=1}^{n} s_m[f]\, W \right) gW \\ &= \sup_{\|gW\|_{L_\infty(I)} \leqslant 1} \frac{1}{n} \sum_{m=1}^{n} \int_I s_m[f]\, gW^2 \\ &= \sup_{\|gW\|_{L_\infty(I)} \leqslant 1} \frac{1}{n} \sum_{m=1}^{n} \int_I f s_m[g]\, W^2\end{aligned}$$

by self-adjointness of s_m (this follows easily from orthogonality). We continue this as

$$\leqslant \sup_{\|gW\|_{L_\infty(I)} \leqslant 1} \frac{1}{n} \sum_{m=1}^{n} \int_I |fW\Psi_n^{-1}| \, |s_m[g] \, W\Psi_n|$$

$$\leqslant \sup_{\|gW\|_{L_\infty(I)} \leqslant 1} \int_I |fW\Psi_n^{-1}| \left\| \frac{1}{n} \sum_{m=1}^{n} |s_m[g] \, W\Psi_n| \right\|_{L_\infty(I)}$$

$$\leqslant C \int_I |fW\Psi_n^{-1}|$$

by our result for $p = \infty$. ▮

Finally, we use weighted interpolation to treat the case $1 < p < \infty$:

Proof of Theorem 4 *for* $1 < p < \infty$. One applies a theorem of E. M. Stein [2, p. 213] on interpolation in weighted spaces. More specifically, if

$$\tau[f] := \frac{1}{n} \sum_{m=1}^{n} s_m[f]$$

and

$$q_0 := 1; \qquad p_0 := 1; \qquad v_0 := W; \qquad u_0 := W\Psi_n^{-1};$$

$$q_1 := \infty; \qquad p_1 := \infty; \qquad v_1 := W\Psi_n; \qquad u_1 := W,$$

we have shown that for $i = 0, 1$ and some C independent of f, n, i

$$\|\tau[f] \, v_i\|_{L_{q_i}(I)} \leqslant C \, \|fu_i\|_{L_{p_i}(I)}$$

and hence if $0 < \theta < 1$ and

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} = 1 - \theta; \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1} = 1 - \theta;$$

$$u := u_0^{1-\theta} u_1^{\theta}; \qquad v := v_0^{1-\theta} v_1^{\theta}$$

then

$$\|\tau[f] \, v\|_{L_q(I)} \leqslant C \, \|fu\|_{L_p(I)}.$$

This is easily reformulated as (21). ▮

Deduction of Corollary 5. Suppose first that $1 \leqslant p < \infty$. Recall that for Freud weights $T \sim 1$ so in $[-a_n, a_n]$,

$$\psi_n = \frac{\phi_n + T(a_n)^{-1}}{\sqrt{\phi_n}} \sim \frac{1}{\sqrt{\phi_n}} \geqslant C$$

$$\Rightarrow \Psi_n = \max\{\psi_n^{1/2}, \psi_n^{2/3}\} \sim \psi_n^{2/3} \sim \phi_n^{-1/3}.$$

Moreover, $\Psi_n \geqslant C\phi_n^{-1/3}$ in $I \backslash [-a_n, a_n]$. Then

$$\left\| \left(\frac{1}{n} \sum_{m=1}^{n} s_m[f] \right) W\phi_n^{-(1-1/p)/3} \right\|_{L_p(I)}$$

$$\leqslant C \left\| \left(\frac{1}{n} \sum_{m=1}^{n} s_m[f] \right) W\Psi_n^{1-1/p} \right\|_{L_p(I)}$$

$$\leqslant C \, \| fW\Psi_n^{-1/p} \|_{L_p(I)} \leqslant C \, \| fW\phi_n^{1/(3p)} \|_{L_p(I)}.$$

Here we have used (21). The case $p = \infty$ is easier. ∎

3. ESTIMATE OF AN ORTHONORMAL POLYNOMIAL SUM

In this section, we prove:

THEOREM 9. *Let $W \in \mathscr{F}$ and assume that for some $\beta > 0$,*

$$\frac{\alpha_n}{a_n} = \frac{1}{2} + O(n^{-\beta}).$$

Then for $n \geqslant 1$ and $x \in \mathbb{R}$,

$$\sum_{m=1}^{n} (p_{m+1} - p_{m-1})^2 (x) \, W^2(x)$$

$$\leqslant C \frac{n}{a_n} \phi_n(x)^{\min\{3/2, 1/2 + 3\beta/2, 3/4(1+\beta)\}}. \qquad (66)$$

We begin with a simple consequence of the recurrence relation:

LEMMA 10.

$$\sum_{m=1}^{n} \alpha_m (p_m - p_{m-1})^2 (x)$$
$$= \sum_{m=1}^{n-1} p_m^2(x)(\alpha_m + \alpha_{m+1} - x) + p_0^2(x)(\alpha_1 - x)$$
$$+ \alpha_n p_n(x)(p_n - p_{n-1})(x). \tag{67}$$

Proof. Recall the recurrence relation

$$xp_{m-1}(x) = \alpha_m p_m(x) + \alpha_{m-1} p_{m-2}(x).$$

Multiplying this by $p_{m-1}(x)$ and adding for $m = 1, 2, ..., n$ gives

$$x \sum_{m=1}^{n} p_{m-1}^2(x) = \sum_{m=1}^{n} \alpha_m p_m(x)\, p_{m-1}(x) + \sum_{m=1}^{n} \alpha_{m-1} p_{m-2}(x)\, p_{m-1}(x).$$

Changing the index of summation from m to $m-1$ in the sum on the left and the second sum on the right gives

$$x \sum_{m=0}^{n-1} p_m^2(x) = 2 \sum_{m=1}^{n} \alpha_m p_m(x)\, p_{m-1}(x) - \alpha_n p_n(x)\, p_{n-1}(x),$$

recall $p_{-1} = 0$. Then

$$\sum_{m=1}^{n} \alpha_m (p_m - p_{m-1})^2 (x)$$
$$= \sum_{m=1}^{n} \alpha_m p_m^2(x) + \sum_{m=1}^{n} \alpha_m p_{m-1}^2(x) - 2 \sum_{m=1}^{n} \alpha_m p_m(x)\, p_{m-1}(x)$$
$$= \sum_{m=1}^{n-1} (\alpha_m + \alpha_{m+1})\, p_m^2(x) + \alpha_n p_n^2(x)$$
$$+ \alpha_1 p_0^2(x) - x \sum_{m=0}^{n-1} p_m^2(x) - \alpha_n p_n(x)\, p_{n-1}(x).$$

Then (67) follows. ∎

Surprisingly the most troublesome term on the right-hand side of (67) is the third term. This is handled in the following lemma: there and in the sequel, we assume that $W \in \mathcal{F}$, that (28) holds, and we shall use the estimates [9, Cor. 1.4, p. 467]

$$|p_n W|\,(x) \leqslant C a_n^{-1/2} \phi_n^{-1/4}(x), \qquad x \in \mathbb{R}, \quad n \geqslant 1 \tag{68}$$

and [9, Lemma 5.2(a), p. 478]

$$\frac{a_{2n}}{a_n} \geqslant C > 1, \qquad n \geqslant 1. \tag{69}$$

LEMMA 11. *For* $x \in [0, a_n]$,

$$|p_n - p_{n-1}|\,(x)\,W(x) \leqslant C n^{\max\{0,\,(1-\beta)/2\}} a_n^{-1/2} \phi_n^{1/4}(x). \tag{70}$$

Proof. We consider two ranges of x:

(i) $x \in [0, \frac{1}{2}a_n]$
Here $\phi_n(x) \sim 1$ and the desired estimate follows from (68).

(ii) $x \in (\frac{1}{2}a_n, a_n]$
We use the Dombrowski–Fricke identity [4, 5, 22] in the form

$$\begin{aligned}\Gamma_n(x) &:= \frac{1}{\alpha_n^2} \sum_{k=0}^{n-1} (\alpha_{k+1}^2 - \alpha_k^2)\, p_k^2(x)\\ &= (p_n - p_{n-1})^2\,(x) + 2p_{n-1}(x)\,p_n(x)\left(1 - \frac{x}{2\alpha_n}\right).\end{aligned}$$

This gives

$$\begin{aligned}&\Gamma_n(x)\,W^2(x)\\ &\quad= ((p_n - p_{n-1})\,W)^2\,(x) + 2(p_{n-1}p_nW^2)(x)\left(\left[1 - \frac{x}{a_n}\right] + O(n^{-\beta})\right)\\ &\quad= ((p_n - p_{n-1})\,W)^2\,(x) + O(a_n^{-1}\phi_n(x)^{1/2})\\ &\qquad + O(a_n^{-1}\phi_n(x)^{-1/2}\,n^{-\beta}).\end{aligned} \tag{71}$$

Here we have used (68), (56) and our hypothesis (28). Next, that hypothesis gives for $0 \leqslant k \leqslant n-1$,

$$\begin{aligned}\alpha_{k+1}^2 - \alpha_k^2 &= \alpha_{k+1}^2\left(1 - \left[\frac{a_k}{a_{k+1}}\right]^2\left[\frac{\alpha_k/a_k}{\alpha_{k+1}/a_{k+1}}\right]^2\right)\\ &= \alpha_{k+1}^2\left(1 - \left[1 + O\left(\frac{1}{k+1}\right)\right]^2 [1 + O((k+1)^{-\beta})]^2\right)\\ &\leqslant C a_n^2 (k+1)^{-\min\{1,\,\beta\}}.\end{aligned}$$

Here we have used not only (28) but also (54) (recall $T \sim 1$ for $W \in \mathscr{F}$). Then from (51), we obtain

$$\Gamma_n(x) \leqslant C \sum_{k=0}^{n-1} (k+1)^{-\min\{1,\,\beta\}}\, p_k^2(x).$$

Recall that $x > \frac{1}{2}a_n$. Now from (69), there exists ε_1 independent of n, such that for large enough n,

$$\tfrac{1}{2}a_n \geqslant a_{2\varepsilon_1 n}.$$

We then use (58) of Lemma 8 applied to W^2 rather than W to deduce that

$$\begin{aligned} W^2(x) \sum_{k=0}^{[\varepsilon_1 n]-1} (k+1)^{-\min\{1,\beta\}}\, p_k^2(x) &\leqslant W^2(x) \sum_{k=0}^{[\varepsilon_1 n]-1} p_k^2(x) \\ &= W^2(x)\, \lambda_{[\varepsilon_1 n]}^{-1}(W^2, x) \\ &\leqslant C_1 \exp(-C_2 n) \sup_{t\in\mathbb{R}} W^2(t)\, \lambda_{[\varepsilon_1 n]}^{-1}(W^2, t) \\ &\leqslant C_3 \exp(-C_4 n). \end{aligned}$$

Next,

$$\begin{aligned} W^2(x) \sum_{k=[\varepsilon_1 n]}^{n-1} (k+1)^{-\min\{1,\beta\}}\, p_k^2(x) &\leqslant C n^{-\min\{1,\beta\}}\, W^2(x)\, \lambda_n^{-1}(W^2, x) \\ &\leqslant C n^{-\min\{1,\beta\}} \frac{n}{a_n} \phi_n(x)^{1/2}, \end{aligned}$$

recall that $\psi_n \sim \phi_n^{-1/2}$ for Freud weights. Thus, the last two estimates yield

$$\Gamma_n(x)\, W^2(x) \leqslant C n^{-\min\{1,\beta\}} \frac{n}{a_n} \phi_n(x)^{1/2} \tag{72}$$

and hence from (71),

$$\begin{aligned} &((p_n - p_{n-1})\, W)^2\, (x) \\ &\quad \leqslant C\left[n^{-\min\{1,\beta\}} \frac{n}{a_n} \phi_n(x)^{1/2} + a_n^{-1} \phi_n(x)^{-1/2}\, n^{-\beta} \right]. \end{aligned} \tag{73}$$

Now by definition of ϕ_n,

$$n^{-1} \leqslant C\phi_n^{3/2} \tag{74}$$

and it then follows that the first term in the right-hand side of (73) is the larger one (apart from a constant), so we obtain (70). ∎

For future use, we record the estimate effectively proved in the above lemma: for $\Delta > 0$,

$$W^2(x) \sum_{k=0}^{n-1} (k+1)^{-\Delta} p_k^2(x) \leqslant Cn^{1-\Delta} a_n^{-1} \phi_n(x)^{1/2} \qquad x \in [\tfrac{1}{2}a_n, a_n]. \quad (75)$$

The next step in the proof of Theorem 9 is:

LEMMA 12. *For* $x \in [0, a_n]$,

$$\sum_{m=1}^{n-1} \alpha_m (p_m - p_{m-1})^2 (x) W^2(x) \leqslant n\phi_n(x)^{\min\{3/2,\, 1/2 + (3/2)\beta,\, 3/4(\beta+1)\}}. \quad (76)$$

Proof. For $x \in [0, \frac{1}{2}a_n]$, the estimate follows easily from (51), (52) since $\phi_n(x) \sim 1$. We now assume that $x \in [\frac{1}{2}a_n, a_n]$. We use (28). Now for $m \leqslant n-1$,

$$\alpha_m + \alpha_{m+1} - x \leqslant \frac{a_m}{2} + \frac{a_{m+1}}{2} - x + O\left(\frac{a_{m+1}}{m^\beta}\right) \leqslant a_n - x + O\left(\frac{a_n}{m^\beta}\right),$$

so

$$\begin{aligned} W^2(x) &\sum_{m=1}^{n-1} p_m^2(x)(\alpha_m + \alpha_{m+1} - x) \\ &\leqslant (a_n - x) W^2(x) \lambda_n^{-1}(W^2, x) + Ca_n W^2(x) \sum_{m=1}^{n-1} \frac{p_m^2(x)}{m^\beta} \\ &\leqslant Cn\phi_n(x)^{3/2} + Cn^{1-\beta}\phi_n(x)^{1/2} \end{aligned}$$

by (52), (75). Using (74), we continue this as

$$W^2(x) \sum_{m=1}^{n-1} p_m^2(x)(\alpha_m + \alpha_{m+1} - x) \leqslant Cn\phi_n(x)^{\min\{3/2,\, 1/2 + (3/2)\beta\}}. \quad (77)$$

Next, combining (68) and (70) gives

$$\begin{aligned} W^2(x)\, \alpha_n\, |p_n(x)(p_n - p_{n-1}(x))| &\leqslant Cn^{\max\{0,\, (1-\beta)/2\}} \leqslant Cn \cdot n^{-\min\{1,\, (\beta+1)/2\}} \\ &\leqslant Cn\phi_n(x)^{(3/2)\min\{1,\, (\beta+1)/2\}}. \end{aligned}$$

Finally,

$$W^2(x)\, p_0^2(x)\, |\alpha_1 - x| \leqslant C \leqslant Cn\phi_n(x)^{3/2}.$$

Combining the last three estimates and (67) gives the result. ∎

We turn to

The Proof of Theorem 9. Firstly for $x \in [0, \frac{1}{4}a_n]$,

$$\sum_{m=1}^{n} (p_{m+1} - p_{m-1})^2\,(x)\, W^2(x) \leqslant 4\lambda_{n+2}^{-1}(W^2, x)\, W^2(x) \leqslant C\frac{n}{a_n}$$

and then (66) follows as $\phi_n(x) \sim 1$. We now assume that $x \in [\frac{1}{4}a_n, a_n]$. Let $\varepsilon_0 \in (0, \frac{1}{2})$. Since

$$(p_{m+1} - p_{m-1})^2 \leqslant 2(p_{m+1} - p_m)^2 + 2(p_m - p_{m-1})^2,$$

we obtain from (51) and then Lemma 12 and (56), that

$$\begin{aligned}
&\sum_{m=[\varepsilon_0 n]}^{n} (p_{m+1} - p_{m-1})^2\,(x)\, W^2(x) \\
&\quad\leqslant \frac{C}{a_n}\left[\sum_{m=[\varepsilon_0 n]}^{n} \alpha_{m+1}(p_{m+1} - p_m)^2\,(x)\, W^2(x)\right. \\
&\qquad\left. + \sum_{m=[\varepsilon_0 n]}^{n} \alpha_m(p_m - p_{m-1})^2\,(x)\, W^2(x)\right] \\
&\quad\leqslant C\frac{n}{a_n}\,\phi_n(x)^{\min\{3/2,\ 1/2 + (3/2)\beta,\ 3/4(\beta+1)\}}.
\end{aligned}$$

If we choose ε_0 small enough, then it follows as in the proof of (72) of Lemma 11 that the contribution of the terms with $m < [\varepsilon_0 n]$ is negligible. Thus we have the desired estimate (66) for $x \in [0, a_n]$ and hence for all $x \in [-a_n, a_n]$, recall that $(p_{m+1} - p_{m-1})^2$ is even. To extend the estimate to the whole real line, one uses the same trick as in the proof of Theorem 4 for $p = \infty$: one approximates powers of ϕ_n by polynomials R_n of degree $O(\delta_n^{-1/2}) = O(n^{1/3})$, and then uses infinite-finite range inequalities. ∎

We shall actually apply not Theorem 9, but a simple consequence thereof:

COROLLARY 13. *Let $W \in \mathcal{F}$ and assume that for some $\beta > 0$,*

$$\frac{\alpha_n}{a_n} = \frac{1}{2} + O(n^{-\beta}).$$

Then for $n \geqslant 1$ and $x \in \mathbb{R}$,

$$\sum_{m=1}^{n} \left(p_{m+1} - \frac{\alpha_m}{\alpha_{m+1}} p_{m-1} \right)^2 (x)\, W^2(x)$$

$$\leqslant C \frac{n}{a_n} \phi_n(x)^{\min\{3/2,\, 1/2 + 3\beta/2,\, 3/4(1+\beta)\}}. \tag{78}$$

Proof. We have

$$\left(p_{m+1} - \frac{\alpha_m}{\alpha_{m+1}} p_{m-1} \right)^2 \leqslant 2(p_{m+1} - p_{m-1})^2 + 2\left(1 - \frac{\alpha_m}{\alpha_{m+1}}\right)^2 p_{m-1}^2$$

$$\leqslant 2(p_{m+1} - p_{m-1})^2 + Cm^{-2\min\{1,\beta\}} p_{m-1}^2.$$

Here for $x \in [\frac{1}{2}a_n, a_n]$, (75) gives

$$W^2(x) \sum_{m=1}^{n} m^{-2\min\{1,\beta\}} p_{m-1}^2(x) \leqslant Cn^{1-2\min\{1,\beta\}} a_n^{-1} \phi_n(x)^{1/2}$$

$$\leqslant C \frac{n}{a_n} \phi_n(x)^{3\min\{1,\beta\} + 1/2}$$

and the rest of the details follow as before. ∎

4. PROOF OF THEOREM 6

Throughout, we assume the hypotheses of Theorem 6. We shall also assume that the sequence ρ_n decays to 0 no faster than some negative power of n. The proof is based on:

An alternative estimate for $s_m[F_n(\cdot)(x - \cdot)](x)$. The alternative estimate involves a simple change of indices in summation, that has been employed several times before (for example in [17]); we do not know who first used it. Recall the notation (1), (44) and the abbreviation $b_m = b_m(F_n)$. Then

$$\begin{aligned}
&\frac{1}{n}\sum_{m=1}^{n} s_m[F_n(\cdot)(x-\cdot)](x)\\
&\quad=\frac{1}{n}\left[\sum_{m=1}^{n}\alpha_m p_m b_{m-1}-\sum_{m=1}^{n}\alpha_m p_{m-1}b_m\right]\\
&\quad=\frac{1}{n}\left[\sum_{m=0}^{n-1}\alpha_{m+1} p_{m+1} b_{m}-\sum_{m=1}^{n}\alpha_m p_{m-1}b_m\right]\\
&\quad=\frac{1}{n}\left[\sum_{m=1}^{n-1} b_m(\alpha_{m+1} p_{m+1}-\alpha_m p_{m-1})+\alpha_1 p_1 b_0-\alpha_n p_{n-1} b_n\right]\\
&\quad:= T^{(1)}+T^{(2)}+T^{(3)}.
\end{aligned} \tag{79}$$

Estimation of $T^{(1)}$. Here

$$\begin{aligned}
|T^{(1)}| &= \frac{1}{n}\left|\sum_{m=1}^{n-1} b_m(\alpha_{m+1} p_{m+1}-\alpha_m p_{m-1})\right|\\
&\leqslant \frac{1}{n}\sqrt{\sum_{m=1}^{n-1} b_m^2}\ \sqrt{\sum_{m=1}^{n-1}(\alpha_{m+1} p_{m+1}-\alpha_m p_{m-1})^2}\\
&\leqslant C\,\|fW\|_{L_\infty(\mathbb{R})}\sqrt{\frac{a_n^2}{n^2\rho_n}}\ \sqrt{\sum_{m=1}^{n}\left(p_{m+1}-\frac{\alpha_m}{\alpha_{m+1}}\,p_{m-1}\right)^2},
\end{aligned}$$

exactly as in the de la Vallee Poussin estimate for $s_m[F_n(\cdot)(x-\cdot)](x)$ (see (44–45)). Using Corollary 13, we continue this as

$$|T^{(1)}W|\,(x) \leqslant C\,\|fW\|_{L_\infty(\mathbb{R})}\sqrt{\frac{a_n}{n\rho_n}}\sqrt{\phi_n(x)^{\min\{3/2,\ 1/2+3\beta/2,\ 3/4(1+\beta)\}}},\qquad x\in\mathbb{R}. \tag{80}$$

Estimation of $T^{(2)}$. Next,

$$|b_0| = \left|\int_I F_n p_0 W^2\right| \leqslant \|fW\|_{L_\infty(\mathbb{R})}\,\gamma_0\int_{|t-x|\geqslant\rho_n}\frac{W(t)}{|t-x|}\,dt.$$

We consider separately two ranges of x:

(I) x such that $a_n \phi_n(x) \geqslant 1$.
Then we estimate

$$
\begin{aligned}
|b_0| &\leqslant \|fW\|_{L_\infty(\mathbb{R})} \gamma_0 \left(W(0) \int_{\rho_n \leqslant |t-x| < a_n \phi_n(x)} \frac{dt}{|t-x|} \right. \\
&\quad \left. + \frac{1}{a_n \phi_n(x)} \int_{|t-x| \geqslant a_n \phi_n(x)} W(t)\, dt \right) \\
&\leqslant C \|fW\|_{L_\infty(\mathbb{R})} \left(\log^+ \left(\frac{a_n \phi_n(x)}{\rho_n} \right) + 1 \right).
\end{aligned}
$$

Here we set

$$\log^+ t := \max\{0, \log t\}.$$

(II) x such that $a_n \phi_n(x) < 1$
Then

$$\left| 1 - \frac{x}{a_n} \right| < \frac{1}{a_n} < \frac{1}{2}$$

for large enough n, so that $|t-x| \leqslant 1 \Rightarrow W(t)$ is geometrically small:

$$|t-x| \leqslant 1 \Rightarrow W(t) \leqslant \exp(-C_1 n).$$

(See [9, Lemma 5.1(c), p. 477]). Then we estimate

$$
\begin{aligned}
|b_0| &\leqslant \|fW\|_{L_\infty(\mathbb{R})} \gamma_0 \left(\exp(-C_1 n) \log^+ \left(\frac{1}{\rho_n} \right) + \int_{|t-x| \geqslant 1} W(t)\, dt \right) \\
&\leqslant C \|fW\|_{L_\infty(\mathbb{R})} \left(\log^+ \left(\frac{a_n \phi_n(x)}{\rho_n} \right) + 1 \right).
\end{aligned}
$$

(Recall our hypothesis $\rho_n \geqslant n^{-C}$). Thus we have this estimate in all cases and hence

$$
\begin{aligned}
|T^{(2)} W|\,(x) &= \frac{1}{n} |\alpha_1 p_1(x)\, b_0|\, W(x) \\
&\leqslant \frac{C}{n} \|fW\|_{L_\infty(\mathbb{R})} \left(\log^+ \left(\frac{a_n \phi_n(x)}{\rho_n} \right) + 1 \right). \qquad (81)
\end{aligned}
$$

Estimation of $T^{(3)}$. It is more difficult to estimate b_n:

$$\begin{aligned}|b_n| &= \left|\int_I F_n p_n W^2\right| \\ &\leqslant C\,\|fW\|_{L_\infty(\mathbb{R})}\left(a_n^{-1/2}\int_{\rho_n\leqslant|t-x|\leqslant(1/4)\,a_n}\frac{\phi_n(t)^{-1/4}}{|t-x|}\,dt\right. \\ &\quad\left.+a_n^{-1}\int_{|t-x|>(1/4)\,a_n}|p_nW|\,(t)\,dt\right) \\ &\leqslant C\,\|fW\|_{L_\infty(\mathbb{R})}\left(a_n^{-1/2}\int_{\rho_n\leqslant|t-x|\leqslant(1/4)\,a_n}\frac{\phi_n(t)^{-1/4}}{|t-x|}\,dt+a_n^{-1/2}\right). \qquad (82)\end{aligned}$$

Here we have used an estimate for the L_1 norm of p_nW from [15, Thm. 1, p. 44]. In subsequent estimation, we consider $x\geqslant 0$, and consider two subcases:

(I) $x\in[0, 1/4a_n]$
Here $|t-x|\leqslant\frac{1}{4}a_n\Rightarrow|t|\leqslant\frac{1}{2}a_n$, so that $\phi_n(t)\sim 1$ and we obtain

$$|b_n|<C\,\|fW\|_{L_\infty(\mathbb{R})}\,a_n^{-1/2}\left[\log^+\left(\frac{a_n}{4\rho_n}\right)+1\right]. \qquad (83)$$

(II) $x\in[\frac{1}{4}a_n, a_n(1-n^{-2/3})]$
Here $|t-x|\leqslant\frac{1}{4}a_n\Rightarrow t\geqslant 0$ and $\phi_n(t)\sim 1-(t/a_n)$ so that

$$\begin{aligned}&\int_{\rho_n\leqslant|t-x|\leqslant(1/4)\,a_n}\frac{\phi_n(t)^{-1/4}}{|t-x|}\,dt \\ &\sim\int_{\rho_n\leqslant|t-x|<(1/4)\,a_n}\frac{\left|1-\dfrac{t}{a_n}\right|^{-1/4}}{a_n\left|\left(1-\dfrac{t}{a_n}\right)-\left(1-\dfrac{x}{a_n}\right)\right|}\,dt \\ &=\left(1-\frac{x}{a_n}\right)^{-1/4}\int_{(\rho_n/a_n(1-(x/a_n)))\leqslant|s-1|\leqslant(1/4(1-(x/a_n)))}\frac{|s|^{-1/4}}{|1-s|}\,ds \\ &\leqslant C\left(1-\frac{x}{a_n}\right)^{-1/4}\left[\log^+\frac{a_n\left(1-\dfrac{x}{\rho_n}\right)}{\rho_n}+1\right],\end{aligned}$$

by first the substitution $1-(t/a_n)=s(1-(x/a_n))$ and then some straightforward estimation. Together with our estimates (82), (83), this shows that for all $x\in[0, a_n(1-n^{2/3})]$,

$$|b_n|\leqslant C\,\|fW\|_{L_\infty(\mathbb{R})}\,a_n^{-1/2}\phi_n(x)^{-1/4}\left[\log^+\frac{a_n\phi_n(x)}{\rho_n}+1\right].$$

Then for $|x|\leqslant a_n(1-n^{-2/3})$,

$$\begin{aligned}|T^{(3)}W|\,(x)&=\frac{1}{n}\,|\alpha_n p_{n-1}(x)\,W(x)\,b_n|\\&\leqslant\frac{C}{n}\,\|fW\|_{L_\infty(\mathbb{R})}\,\phi_n(x)^{-1/2}\left[\log^+\frac{a_n\phi_n(x)}{\rho_n}+1\right].\end{aligned}$$

We obtain from (79)–(81) and this last estimate that for $|x|\leqslant a_n(1-n^{-2/3})$,

$$\begin{aligned}&\left|\frac{1}{n}\sum_{m=1}^{n}s_m[F_n(\cdot)(x-\cdot)](x)\right|W(x)\\&\quad\leqslant C\,\|fW\|_{L_\infty(\mathbb{R})}\left\{\sqrt{\frac{a_n}{n\rho_n}}\,\phi_n(x)^{\min\{3/4,\,1/4+3\beta/4,\,3/8(1+\beta)\}}\right.\\&\qquad\left.+n^{-1}\phi_n^{-1/2}(x)\left[\log^+\frac{a_n\phi_n(x)}{\rho_n}+1\right]\right\}.\end{aligned}\tag{84}$$

We turn to

The Proof of Theorem 6. Combining (36), (48), (52), (84) gives for $|x|\leqslant a_n(1-n^{-2/3})$,

$$\begin{aligned}\Gamma:=\frac{1}{n}&\left|\sum_{m=1}^{n}s_m[f](x)\right|W(x)\\&\leqslant C\,\|fW\|_{L_\infty(\mathbb{R})}\left\{\begin{aligned}&\sqrt{\frac{a_n}{n\rho_n}}\,\phi_n(x)^{\min\{3/4,\,1/4+3\beta/4,\,3/8(1+\beta)\}}+n^{-1}\phi_n^{-1/2}(x)\\&\qquad\times\log^+\left[\frac{a_n\phi_n(x)}{\rho_n}+1\right]\\&\qquad+\frac{n\rho_n}{a_n}\,\phi_n^{1/4}(x)\,\max_{|t-x|\leqslant\rho_n}\phi_n^{1/4}(t)\end{aligned}\right\}.\end{aligned}\tag{85}$$

(Recall that for Freud weights, $\psi_n \sim \phi_n^{-1/2}$). Now fix x such that $|x| \leqslant a_n(1-n^{-2/3})$, fix $\Delta \in [0, \frac{1}{2})$ and set

$$\rho_n := \frac{a_n}{n}\, \phi_n^{\Delta}(x) \leqslant C \frac{a_n}{n}\, \phi_n^{-1/2}(x) \leqslant C \frac{a_n}{n}\, \psi_n(x). \tag{86}$$

Then (53) shows that for $|t-x| \leqslant \rho_n$, $\phi_n(t) \sim \phi_n(x)$. So (85) becomes

$$\begin{aligned} \Gamma \leqslant{} & C\, \|fW\|_{L_\infty(\mathbb{R})} \\ & \times \left\{ \begin{aligned} & \phi_n(x)^{-\Delta/2 + \min\{3/4,\, 1/4+3\beta/4,\, 3/8(1+\beta)\}} \\ & \quad + n^{-1}\phi_n^{-1/2}(x)[\log^+[n\phi_n^{1-\Delta}(x)] + 1] + \phi_n^{\Delta+1/2}(x) \end{aligned} \right\}. \end{aligned} \tag{87}$$

The ratio of the second and third terms in the last right-hand side is

$$\begin{aligned} & n^{-1}\phi_n^{-1-\Delta}(x)[\log^+[n\phi_n^{1-\Delta}(x)] + 1] \\ & \quad \leqslant C \frac{\log n}{n}\, \phi_n^{-1-\Delta}(x) \leqslant C \frac{\log n}{n}\, (n^{-2/3})^{-1-\Delta} = o(1) \end{aligned}$$

as $\Delta < \frac{1}{2}$. It follows that the second term in the right-hand side of (87) is bounded by a constant times the third. Finally, we deduce for $|x| \leqslant a_n(1-n^{-2/3})$,

$$\Gamma \leqslant C\, \|fW\|_{L_\infty(\mathbb{R})}\, \phi_n(x)^{\min\{-\Delta/2 + \min\{3/4,\, 1/4+3\beta/4,\, 3/8(1+\beta)\},\, \Delta+1/2\}}$$

Choosing

$$\Delta := \frac{2}{3} \min \left\{ \frac{3}{4}, \frac{1}{4} + \frac{3\beta}{4}, \frac{3}{8}(1+\beta) \right\} - \frac{1}{3} \in \left[0, \frac{1}{6}\right]$$

gives for $|x| \leqslant a_n(1-n^{-2/3})$,

$$\begin{aligned} & \frac{1}{n} \left| \sum_{m=1}^{n} s_m[f](x) \right| \\ & \quad \leqslant C\, \|fW\|_{L_\infty(\mathbb{R})}\, \phi_n(x)^{2/3 \min\{3/4,\, 1/4+3\beta/4,\, 3/8(1+\beta)\} + 1/6}. \end{aligned}$$

We extend this estimate to the whole real line exactly as in the proof of Theorem 4. Then we obtain (31) for $p = \infty$. The extension to $p \in [1, \infty)$ follows as in the proof of Theorem 4 for that range of p. ∎

Proof of Corollary 7. As we have noted, Kriecherbauer and McLaughlin proved that (28) holds for $W = W_\alpha$ with $\beta = \min\{\alpha, 2\}$. Then $\kappa = 2/3$ in (29) and the result follows. ∎

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